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OPTIMAL INVESTMENT, FINANCING, AND DIVIDENDS
A Stackelberg Differential Game*

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Over a finite planning period, a firm has two shareholders who trade shares at a fixed price. External transactions are disregarded. The total amount of shares is fixed and the majority shareholder decides on the rate of dividend payout. Each shareholder maximizes a profit functional comprised of total earnings from share transactions plus dividends, and capital gains at the horizon date. The shareholders are subject to personal taxation on dividends and capital gains. Decisions on investments and borrowing/lending are made by a manager who maximizes accumulated profits after corporate taxation.

The problem is modelled as an open-loop Stackelberg differential game such that the manager is the leader; the shareholders are followers, playing a Nash game. The latter game is analyzed by using standard techniques of optimal control theory. The analysis off the manager's problem is done by using a path-connecting procedure.

1. Introduction

This paper deals with the influence of corporate as well as personal taxation on the optimal investment, financing, and dividend policies of a firm. Recent contributions in this area include Ludwig (1978), Ylä-Liedenpohja (1978), van Loon (1983), and van Schijndel (1986a, b, 1987). See also the survey article by Lesourne and Leban (1982). These studies, however, assume no separation of ownership and management, i.e., the shareholders are also the managers of the firm. The main topic in this research is the determination of optimal policies for capital investments, dividends, and debt. In this connection, the question of corporate taxation is important and was the first to receive

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attention in the literature. Later works [for instance, van Schijndel (1986a, b, 1987)] also considered the impact of different personal tax rates of the shareholders.

The purpose of this paper is to relax the assumption of nonseparated ownership and management. Within the framework of a financial model of the firm we consider a company with a manager and (for mathematical convenience) only two shareholders. The latter have different personal tax rates which, in turn, differ from the corporate tax rate. The manager controls the investment rate and is in charge of debt management too. The shareholders control the rate of dividend pay-out and can buy and sell shares from each other. No emissions of new stock are undertaken during the time period under consideration and there are no external transactions with shares. Thus, in this respect, the company is viewed as a closed system.

To model a situation with multiple decision-makers we apply the theory of differential games. Various conceptual problems arise here, and we shall briefly discuss some of them. The main body of the paper is devoted to the analysis of a noncooperative game where the manager is a Stackelberg leader, announcing at the start of the game an investment policy and the stockholders respond rationally (as followers) by choosing a dividend policy as well as the amount of internal trade with shares. Because of the complexity of the model, a closed-form solution is apparently not attainable but a number of qualitative results can be stated.

The paper is organized as follows: in section 2 we establish a differential game model as an open-loop Stackelberg game. This section also contains our reflexions on some conceptual difficulties in the modelling process. In section 3 the Stackelberg game is analyzed; we characterize the structure of optimal policies and discuss their economic implications. Section 4 concludes the paper with a brief summary of our main results and contains also some suggestions for further research.

2. Model formulation

2.1. Preliminaries

In this section we develop a deterministic dynamic model of a corporate firm with a manager (M) and two shareholders (P_1 and P_2, respectively). Let t denote time and $[0, T]$ a planning period of fixed duration. Define:

$K = K(t)$: stock of capital goods. $K(0) = K_0 > 0$ and fixed.
$Y = Y(t)$: debt ($Y > 0$); lending ($Y < 0$).
$\bar{Z}$: common stock at nominal value. $\bar{Z} > 0$ and constant.
$R = R(t)$: cumulative retained earnings. $R(0) = R_0 > 0$ and fixed.

This yields the balance equation

$$K(t) = Y(t) + \bar{Z} + R(t). \tag{1}$$

Notice that we have assumed that issue of new shares is not allowed, i.e., $\bar{Z}$ is constant. Suppose that

- the firm operates under decreasing returns,
- corporate tax is proportional to profit,
- depreciation is proportional to the stock of capital goods,
- borrowing/lending do not carry any transactions costs and can be effected at
  the same rate of interest,
- dividends are paid out in cash (not in shares).

The flow of retained earnings is given by

$$E = (1 - f)(G(K) - aK - rY) - D, \tag{2}$$

where

$D = D(t)$: dividend payout rate,
$E = E(t)$: retained earnings rate,
$G = G(K(t))$: gross revenue. $G > 0$, $G' > 0$, $G'' < 0$,
a: depreciation rate. $a > 0$ and constant,
f: corporate tax rate. $0 < f < 1$, constant,
r: interest rate on debt/lending. $r > 0$ and constant.

Cumulative retained earnings are given by

$$R(t) = R_0 + \int_0^t \mathrm{E}(s)\,\mathrm{d}s, \tag{3}$$

and (2)–(3) yields our first state equation:

$$\dot{R} = E = (1 - f)(G(K) - aK - rY) - D. \tag{4}$$

Let $I = I(t)$ denote the rate of (gross) investment in capital stock. Then our second state equation becomes

$$\dot{K} = I - aK. \tag{5}$$

Now we turn to the division of common stock, $\bar{Z}$, between the shareholders P_1 and P_2:

$Z_i = Z_i(t)$: P_i's part of $\bar{Z} \cdot Z_i(0) = Z_{i0} > 0$ and fixed, such that $\bar{Z} = Z_1 + Z_2$.

Since $\bar{Z}$ is constant, we obtain

$$\dot{Z}_i = -\dot{Z}_j \tag{6}$$

In (6), and in the sequel, we have $i = 1, 2$; $j = 1, 2$; $i \neq j$. Suppose that P_1 and P_2 may trade shares but not sell to/buy from others. Define

$B_i = B_i(t)$: purchase rate of P_i,
$S_i = S_i(t)$: selling rate of P_i.

From (6) we obtain

$$\dot{Z}_i = B_i - S_i = S_j - S_i. \tag{7}$$

Under the assumptions we only need one of the equations in (7) and take $\dot{Z}_1 = S_2 - S_1$ as our third state equation. Henceforth we refer to this equation as (7). We shall suppose that the shareholders can agree that, if P_i wants to sell, P_j must buy. (Hence, a shareholder always has the option to leave the company, but no shareholder can be forced to give up a majority position or leave the company). The implication is that only the selling rates are control instruments of the shareholders. We also assume that p, the price at which share transactions take place, is fixed (positive).

To construct the payoffs of the shareholders define:

τ_i: tax rate on personal income for P_i; $0 \leq \tau_i < 1$,
τ_g: tax rate on capital gains for P_i; $0 < \tau_g < 1$,

and suppose that P_1 has a high personal tax rate whereas P_2 is in the opposite position. To reflect this let

$$\tau_1 > \tau_g > \tau_2. \tag{8}$$

Dividends are paid out continuously as a fraction of common stock and total dividends amount to $D = C\bar{Z}$, where

$C = C(t)$: dividend payout fraction.

Shareholder P_i receives dividends (before taxation) in amount of $D_i = CZ_i$. Obviously, $D_1 + D_2 = D$.

Assume that each shareholder maximizes his net income stream from share transactions plus dividends, and his share of the firm's equity capital at $t = T$, after capital gains taxation. The payoff functional of P_i becomes

$$J^i = \int_0^T \left[p(S_i - S_j) + (1 - \tau_i)CZ_i \right] \mathrm{d}t + \left[(1 - \tau_g)R(T) + \bar{Z} \right] \frac{Z_i(T)}{\bar{Z}}. \tag{9}$$

The manager maximizes total profits after corporate taxation,

$$J = \int_0^T (1 - f)\left[G(K) - aK - rY \right] \mathrm{d}t.^{1} \tag{10}$$

In what is to follow we assume that the manager is the leader in a Stackelberg game and the shareholders are followers, playing a Nash game vis-a-vis each other.[2] For reasons of tractability we suppose that all players employ open-loop strategies. The pros and cons of such strategies are well known and we shall not pursue the subject any further here. Shareholder P_1 has $Z_{10} > \bar{Z}/2$ which means that he is initially in control of the dividend policy, $C(t)$. We shall assume that P_1 wishes to be in control throughout the game and impose the state constraint.

$$Z_1(t) \geq \bar{Z}/2, \quad \forall t, \tag{11}$$

for which P_1 is responsible. Shareholder P_2 must guarantee satisfaction of

$$Z_1 \leq \bar{Z} \leftrightarrow Z_2 \geq 0, \quad \forall t. \tag{12}$$

[1]We have not incorporated discounting in the payoffs. For the shareholders we can assume a zero discount rate because the possibility of lending money offers the shareholders an alternative investment opportunity with a rate of return equal to r. Hence a shareholder has a time preference rate of $(1 - \tau_i)r$. For the manager it turns out that incorporation of a discount rate does not change qualitatively the results. This is contrary to what is known from models of the same structure but where management and ownership are *not* separated [see, for instance, van Loon (1983)]. The reason is, briefly stated, that the discount rate in such models influences the dividend policy. However, in the present model where management and ownership are separated, the dividend policy is not a control instrument of the manager.

[2]A motivation for the choice of this particular scenario can be found in section 2.2 which also contains some alternative possibilities that might be considered.

and the following control constraints are imposed

$$0 \le C \le C^M \quad (C^M \text{ constant} > 0) \qquad (\text{shareholder } P_1), \tag{13a}$$

$$0 \le S_i \le S^M \quad (S^M \text{ constant} > 0) \qquad (\text{shareholder } P_i),^3 \tag{13b}$$

$$0 \le I \le I^M \quad (I^M \text{ constant} > 0) \qquad (\text{manager}). \tag{13c}$$

The state variable K must satisfy the obvious constraint

$$K \ge 0, \tag{14}$$

which holds whenever $I \ge 0$. [This is easily seen from (5).] Now, it can be argued [see, for example, van Loon (1983)] that debts must not exceed a certain fraction of equity capital, that is,

$$Y \le k(R + \bar{Z}), \qquad k = \text{constant} > 0.$$

This inequality is equivalent to

$$K \le (1 + k)(R + \bar{Z}). \tag{15}$$

Using (14) yields

$$K = Y + \bar{Z} + R \ge 0 \;\Rightarrow\; -Y \le \bar{Z} + R,$$

which means that, in case of lending (i.e., $Y < 0$), the total amount lent cannot exceed total equity capital. Notice that satisfaction of (14) guarantees that this will actually hold. Since debt management is a responsibility of the manager, we assume that he guarantees satisfaction of (15).

In summary, we have posed an open-loop Stackelberg differential game with the following components:

$$\dot{K} = I - aK, \qquad K(0) = K_0 > 0,$$

$$\dot{R} = (1 - f)\big[G(K) - aK - r(K - \bar{Z} - R)\big] - C\bar{Z}, \qquad R(0) = R_0 > 0,$$

$$\dot{Z}_1 = S_2 - S_1, \qquad Z_1(0) = Z_{10} > \bar{Z}/2,$$

[3] The assumption of a common upper bound on the S_i's is motivated partly by mathematical convenience, partly by lack of reason for supposing the opposite.

and P_1, P_2 play – for a fixed $I(t)$ – the Nash game:

$$P_1: \quad \max_{\substack{0 \le C \le C^M \\ 0 \le S_1 \le S^M}} \left\{ J^1 = \int_0^T [p(S_1 - S_2) + (1-\tau_1)CZ_1]\,\mathrm{d}t + \left[(1-\tau_g)R(T) + \bar{Z}\right] Z_1(T)/\bar{Z} \right\}$$

subject to (4), (5), (7), and $Z_1 - \bar{Z}/2 \ge 0$,

$$P_2: \quad \max_{0 \le S_2 \le S^M} \left\{ J^2 = \int_0^T [p(S_2 - S_1) + (1-\tau_2)C(\bar{Z} - Z_1)]\,\mathrm{d}t + \left[(1-\tau_g)R(T) + \bar{Z}\right](\bar{Z} - Z_1(T))/\bar{Z} \right\}$$

subject to (4), (5), (7), and $\bar{Z} - Z_1 \ge 0$.

M solves the optimization problem

$$M: \quad \max_{0 \le I \le I^M} \left\{ J = \int_0^T (1-f)[G(K) - aK - r(K - \bar{Z} - R)]\,\mathrm{d}t \right\}$$

subject to (4), (5), (7), $K \le (1+k)(R + \bar{Z})$,

and the followers' rational reactions.

2.2. *Discussion of some open issues*

The model of section 2.1 poses some conceptual difficulties that deserve consideration. In what follows, we discuss two such aspects.

2.2.1. *Hierarchical relationships*

(1) In this paper we assume that ownership and management are divided such that the shareholders have delegated the daily operations of the firm to a manager. The latter (M) decides upon the investment plan, $I(t)$, and the amount of debt/lending, $Y(t)$. [Notice that the variable Y, by using (1), can be eliminated from the model.] We take M as the leader in a Stackelberg game where M at the start of the game announces his control path $I(t)$. The shareholders, P_1 and P_2, are followers and play a Nash game. Technically, we solve (first) the Nash game for P_1 and P_2, who choose C, S_1, and S_2, taking I as given. This results in reaction functions $C(I)$ and $S_i(I)$, where I is

considered a time-varying parameter. Next, we solve the leader's optimal control problem with respect to I.

(2) Bagchi (1984) considers a reverse case, namely a hierarchical game with the shareholders as leaders and the manager as follower. Thus, M decides on investment I, taking C, S_1, and S_2 as given. This yields M's reaction function $I(C, S_1, S_2)$. Next, the Nash game between P_1 and P_2 is played to determine C, S_1, and S_2.

In case (1), the manager is in a stronger position since he can impose his strategy upon the shareholders. Such a scenario could emerge when the manager is authorized to control the firm's investment and debt management, without interference from the shareholders. The latter make their decisions rather passively (in particular, the dividend policy) in view of the announced policy of the manager. In case (2), the shareholders announce their decisions first, and the manager has no other choice than to react rationally upon this.

(3) A three-level hierarchy with, for example, P_1 at the highest level, P_2 at the middle level, and M at the lowest level may also be conceived.

(4) Assuming that the owners manage the firm, the shareholders control C, I (and the selling rates). This one-decision-maker (optimal control) problem has been studied by, for instance, van Loon (1983) and van Schijndel (1987), but in these works no share transactions take place.

2.2.2. *Control of dividend policy*

There are also open questions regarding the determination of the dividend rate, C. If Z_{i0} is greater than $\bar{Z}/2$, P_i initially has the voting power to fix C.[4] However, it may happen that Z_i changes over time as a result of buying and selling, and P_i may lose control of C if, at some instant, $Z_i/\bar{Z}$ goes below one half. Thus, it may happen that C changes from being a control of P_i to being a control of P_j, and perhaps changes back again to P_i at some later instant of time. Such a situation is not well understood in the differential game literature, and it is not clear how to handle it in an appropriate way. Let us briefly look at some proposals.

(A) In this paper we argue that, if P_1 initially controls C, he is reluctant to give up his control. Hence, he wants to satisfy the constraint $Z_1 \geq \bar{Z}/2$ for all t. Formally, we add the constraint to the model and make P_1 responsible for its satisfaction.[5]

[4] We assume that all shares have equal voting rights, although in practice some shares may have limited (or no) voting rights.

[5] If P_2 could decide on the dividend policy, he would pursue a completely different strategy than the one of P_1, due to P_2's low personal tax rate.

(B) Change the dividend term, $C\bar{Z}$, into $C_i\bar{Z}$ and let C_i be a control variable of the majority shareholder. This implies (among other things) that the dynamics for the state variable R [cf. (4)] will switch as majority switches. Such a situation may call for the use of control theory with switching dynamics [Luhmer (1983)].

(C) A formulation with nice properties is the following.[6] Let the dividend payout rate, C, be determined as $C = wu_1 + (1-w)u_2$, where $w = Z_1/\bar{Z}$ and u_1 and u_2 are continuous control variables of P_1 and P_2, respectively. This formulation means that each shareholder's possibility of influencing the dividend percentage depends on his voting strength, expressed by the amount of shares in his possession.

Proposal (B) offers an intuitively appealing and flexible way to deal with the problem of switches in majority, but is technically complicated. Proposal (C) gives a smooth, but not quite realistic, approximation.

3. Analysis of the differential game

3.1. The shareholders' problem

In section 3.1 we solve the Nash game for the shareholders, taking the investment policy $I(t)$ as a fixed time function. First, we determine optimal selling policies, S_1^* and S_2^*, of the shareholders P_1 and P_2, respectively. Next, we prove infeasibility of certain S_1, S_2 combinations and establish an optimal pair (S_1^*, S_2^*) in a general form, taking into account the satisfaction of the state constraints. Finally, the optimal dividend policy, C^*, of shareholder P_1 is characterized.

3.1.1. Optimal selling policies

We use (1) to eliminate the variable Y. For the majority shareholder P_1 define the Hamiltonian H^1 and the Lagrangian L^1 as follows:

$$H^1 = \lambda_0^1[p(S_1 - S_2) + (1-\tau_1)CZ_1] + \lambda_1^1(S_2 - S_1) + \lambda_2^1(I - aK)$$
$$+\lambda_3^1[(1-f)[G(K) - aK - r(K - \bar{Z} - R)] - C\bar{Z}] \qquad (16)$$

where $\lambda_i^1 = \lambda_i^1(t)$ $(i = 1,3)$ are piecewise continuously differentiable costate variables and $\lambda_0^1 = \text{constant} \geq 0$, and

$$L^1 = H^1 + v^1(Z_1 - \bar{Z}/2). \qquad (17)$$

[6] We are indebted to Paul van Loon for this suggestion.

We have adjoined the state variable constraint $Z_1 \geq \bar{Z}/2$ directly to the Hamiltonian by a piecewise continuous multiplier function $v^1 = v^1(t)$. If an optimal solution exists, it satisfies the necessary conditions

$$(C^*, S_1^*) = \underset{\substack{0 \leq C \leq C^M \\ 0 \leq S_1 \leq S^M}}{\arg\max}\, H^1(Z_1^*, K^*, R^*, C, S_1, S_2^*, I, \lambda_0^1, \lambda_1^1, \lambda_2^1, \lambda_3^1). \tag{18}$$

Condition (18) yields

$$\lambda_0^1(1-\tau_1)Z_1^* - \lambda_3^1\bar{Z} \gtreqless 0 \Rightarrow C^* = \begin{cases} C^M, \\ \text{unspecified}, \\ 0, \end{cases} \tag{19}$$

and

$$\lambda_0^1 p - \lambda_1^1 \gtreqless 0 \Rightarrow S_1^* = \begin{cases} S^M, \\ \text{unspecified}, \\ 0. \end{cases} \tag{20}$$

The costate variables and the multipliers satisfy

$$\dot{\lambda}_1^1 = -\lambda_0^1(1-\tau_1)C^* - v^1, \tag{21a}$$

$$\dot{\lambda}_2^1 = a\lambda_2^1 - \lambda_3^1(1-f)[G'(K^*) - a - r], \tag{21b}$$

$$\dot{\lambda}_3^1 = -\lambda_3^1(1-f)r, \tag{21c}$$

$$v^1 \geq 0, \qquad v^1(Z_1^* - \bar{Z}/2) = 0, \tag{22}$$

$$\lambda_1^1(T) = \lambda_0^1[(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z} + \gamma^1, \tag{23a}$$

$$\lambda_2^1(T) = 0, \tag{23b}$$

$$\lambda_3^1(T) = \lambda_0^1(1-\tau_g)Z_1^*(T)/\bar{Z}, \tag{23c}$$

$$\gamma^1 \geq 0, \qquad \gamma^1(Z_1^*(T) - \bar{Z}/2) = 0, \qquad \gamma^1 = \text{constant}. \tag{24}$$

Lemma 1. $\lambda_0^1 > 0$, *and we put* $\lambda_0^1 = 1$.

Proof. Omitted. It can be obtained from the authors upon request.

From (21a) we obtain

$$\dot{\lambda}_1^1 \leq 0, \tag{25}$$

and integration in (21a), using (23a), yields

$$\lambda_1^1 = \left[(1-\tau_g)R^*(T) + \bar{Z}\right]/\bar{Z} + \int_t^T \left[(1-\tau_1)C^*(s) + v^1(s)\right] \mathrm{d}s + \gamma^1, \tag{26}$$

which is positive for all $t \in [0, T]$. Note that λ_1^1 has the interpretation of the shadow price of a unit of Z_1, as assessed by P_1. Hence this shadow price is positive but nonincreasing.

From (20) and (25)–(26) it is obvious that an optimal S_1 policy must be of one of the following types:

(A) $S_1^* \equiv 0$, which occurs if $\lambda_1^1 > p$, $\forall t \in [0, T]$,

(B) $S_1^* = 0$ on $[0, t_1)$ and $S_1^* = S^M$ on $[t_1, T]$, which occurs if $\lambda_1^1 > p$ for $t \in [0, t_1)$, $\lambda_1^1 = p$ at $t = t_1$, and $\lambda_1^1 < p$ for $t \in (t_1, T]$, (27)

(C) $S_1^* \equiv S^M$, which occurs if $\lambda_1^1 < p$, $\forall t \in [0, T]$.

The derivations of the optimal selling policy S_2^* for shareholder P_2 are very much the same as the ones for P_1 and details are omitted. In (28), the multiplier function λ_1^2 is the shadow price of a unit of Z_1, as assessed by P_2. This shadow price is nonpositive and nondecreasing. The policy S_2^* takes one of the following three forms:

(A) $S_2^* \equiv 0$, which occurs if $\lambda_1^2 < -p \leftrightarrow |\lambda_1^2| > p$, $\forall t$,

(B) $S_2^* = 0$ on $[0, t_2)$ and $S_2^* = S^M$ on $[t_2, T]$, which occurs if $\lambda_1^2 < -p$ for $t \in [0, t_2)$, $\lambda_1^2(t_2) = -p$ and $\lambda_1^2 > -p$ for $t \in [t_2, T]$, (28)

(C) $S_2^* \equiv S^M$, which occurs if $\lambda_1^2 > -p \leftrightarrow |\lambda_1^2| < p$, $\forall t$.

Table 1

Summary of conditions for occurrence of S_i^* policies.

Player P_1	Player P_2
(A) $S_1^* \equiv 0$ if $\kappa + \gamma^1 > p$	(A) $S_2^* \equiv 0$ if $\kappa + \gamma^2 > p$
(B) $S_1^* = 0$ on $[0, t_1)$ $S_1^* = S^M$ on $[t_1, T]$ if $(\kappa + \gamma^1 < p) \wedge \left(\kappa + \gamma^1 + \int_0^T [(1-\tau_1)C^*(s) + v^1(s)]\,ds > p\right)$	(B) $S_2^* = 0$ on $[0, t_2)$ $S_2^* = S^M$ on $[t_2, T]$ if $(\kappa + \gamma^2 < p) \wedge \left(\kappa + \gamma^2 + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]\,ds > p\right)$
(C) $S_1^* \equiv S^M$ If $\kappa + \gamma^1 + \int_0^T [(1-\tau_1)C^*(s) + v^1(d)]\,ds < p$	(C) $S_2^* \equiv S^M$ if $\kappa + \gamma^2 + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]\,ds < p$
$\kappa := [(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z}$	

The results so far obtained for the optimal selling policies S_1^* and S_2^* are summarized in tables 1 and 2.[7]

Lemma 2. The regimes depicted in cells (2), (3), (4), *and* (7) *of table* 2 *are infeasible.*

Proof. See appendix 1.

The lemma states that is never optimal for a player to sell at the maximal rate for all t if the other player does not sell at all, and vice versa. The lemma also states that it is never optimal for a player to use a switching policy against a zero selling policy of the other player, and vice versa.

The remaining regimes (6) and (8) [as well as (1) and (9)] are all subsumed under regime (5). Therefore, we confine our interest to regime (5). The switching instants (t_1 and t_2, respectively) are determined by

$$\lambda_1^i(t_i) = \gamma^i + \kappa + \int_{t_i}^T [(1-\tau_i)C^*(s) + v^i(s)]\,ds = p, \qquad i = 1,2. \tag{29}$$

[7]Notice in table 2 that in the zero-selling and in the maximum-selling case, we have $Z_1 = Z_{10}$ for all $t \le t \le T$. Also notice that whenever $S_1^* = S_2^* = S^M$ on an interval, there is *de facto* no trade; the shareholders simply exchange equal amounts of shares which means that the net amount of trade is zero.

Table 2

Summary of (S_1^*, S_2^*) regimes.

P_1 \ P_2	(A) $S_2^* \equiv 0$	(B) $S_2^* = \begin{cases} 0 & \text{on } [0, t_2) \\ S^M & \text{on } [t_2, T] \end{cases}$	(C) $S_2^* \equiv S^M$
(A) $S_1^* \equiv 0$	(1) The zero-selling case ($t_1 = t_2 = T$)	(2) Infeasible	(3) Infeasible
(B) $S_1^* = \begin{cases} 0 & \text{on } [0, t_1) \\ S^M & \text{on } [t_1, T] \end{cases}$	(4) Infeasible	(5) The general case	(6) Case (5) with $t_2 = 0$ ($t_1 > t_2$)
(C) $S_1^* \equiv S^M$	(7) Infeasible	(8) Case (5) with $t_1 = 0$ ($t_1 < t_2$)	(9) The maximum-selling case ($t_1 = t_2 = 0$)

Notice that if $t_1 = t_2$, then $Z_1 \equiv Z_{10}$ as in regimes (1) and (9). The following lemma can be established, implying that if suffices to consider regime (5) for the case of $t_1 < t_2$.

Lemma 3. If in regime (5) *the switching instants are such that* $t_1 > t_2$, *regime* (5) *reduces to regime* (1) *or* (9).

Proof. See appendix 2.

In what follows we look at the situation where the state constraint $Z_1 \geq \bar{Z}/2$ could become binding. For this purpose, consider regime (5) with $t_1 < t_2$ and the inequality

$$Z_{10} - S^M(t_2 - t_1) > \bar{Z}/2 \leftrightarrow S^M(t_2 - t_1) < Z_{10} - \bar{Z}/2, \tag{30}$$

which is satisfied if, for example, Z_{10} is much larger than $\bar{Z}/2$ (i.e., P_1 has initially a comfortable majority), S^M is relatively small, or t_1 is close to t_2. It turns out that the policy S_1 depends on whether (30) is satisfied or not. Obviously, $Z_1 \equiv Z_{10}$ on $[0, t_1)$ and no constraints are binding. On $[t_1, t_2)$ we have $\dot{Z}_1 < 0$ implying that (12) cannot be binding; hence $v^2 = 0$. The same holds true on the interval $[t_2, T]$: $v^2 = 0$ on $[t_2, T]$ and $\gamma^2 = 0$. It may happen, however, that (11) becomes tight for some t in the interval $[t_1, t_2)$. But note that if (11) does not become binding in $[t_1, t_2)$ it never does. For $t \in [t_1, t_2)$ we have

$$Z_1 = Z_{10} - S^M(t - t_1),$$

and suppose that Z_1 hits its lower bound $(\bar{Z}/2)$ at $t = \tilde{t}_1$ $(\tilde{t}_1 > t_1)$. There are two subcases to consider.

(I) If $\tilde{t}_1 \geq t_2$, then $Z_1 > \bar{Z}/2$, $\forall t \in [t_1, t_2]$, and the policy for S_1 is policy (B) given by (27). It is easy to see that this situation occurs if (30) is satisfied.

(II) If $\tilde{t}_1 < t_2$, then $Z_1(\tilde{t}_1) = \bar{Z}/2$. Since we have made P_1 responsible for the satisfaction of the constraint (11), this player must switch from $S_1 = S^M$ to $S_1 = 0$ on the interval $[\tilde{t}_1, t_2)$. This will keep Z_1 equal to its lower bound on $[\tilde{t}_1, t_2)$. When P_2 switches (at $t = t_2$) from $S_2 = 0$ to $S_2 = S^M$, P_1 resumes his policy $S_1 = S^M$. This situation occurs if (30) is not satisfied.

To summarize: for $t_1 < t_2$ and if (30) does not hold, the S_1 policy should be modified such that

$$\begin{aligned}
&S_1^* = 0 \;\Rightarrow\; Z_1 = Z_{10} && \text{on}\,[0, t_1),\\
&S_1^* = S^M \;\Rightarrow\; Z_1 = Z_{10} - S^M(t - t_1) && \text{on}\,[t_1, \tilde{t}_1),\\
&S_1^* = 0 \;\Rightarrow\; Z_1 = \bar{Z}/2 && \text{on}\,[\tilde{t}_1, t_2),\\
&S_1^* = S^M \;\Rightarrow\; Z_1 = \bar{Z}/2 && \text{on}\,[t_2, T].
\end{aligned} \tag{31}$$

We proceed with a characterization of the optimal dividend policy, C^*. Integration in (21c) and using (23c) yields

$$\lambda_3^1 = \exp\{(1-f)r(T-t)\}\left[(1-\tau_g)Z_1^*(T)/\bar{Z}\right], \tag{32}$$

which is positive for all $t \in [0, T]$. The costate λ_3^1 represents the shadow price of a unit of R, as assessed by P_1. Using (21c) yields

$$\dot{\lambda}_3^1 < 0, \qquad \forall t \in [0, T]. \tag{33}$$

Hence the shadow price of a unit of R (evaluated by P_1) is positive but strictly decreasing. We get the following types of C policies:

$$C^* = \begin{cases} C^M & \\ C \in [0, C^M] & \text{if} \quad (1-\tau_1)\dfrac{Z_1}{\overline{Z}} \gtreqless \lambda_3^1 \leftrightarrow \\ 0 & \quad (1-\tau_1)\dfrac{Z_1}{\overline{Z}} \gtreqless (1-\tau_g)\dfrac{Z_1(T)}{\overline{Z}} \\ & \quad \times \exp\{(1-f)r(T-t)\}, \end{cases} \tag{34}$$

where the following lemma can be proved.

Lemma 4. A singular $C(t)$ is infeasible.

Proof. See appendix 3.

Next, we determine which types of dividend policies can occur under the optimal selling policies in regimes (1), (5) ($t_1 < t_2$), and (9). Using (34) and table 2 we obtain the following results:

Regime (1) and (9): $C^* \equiv 0$.

Regime (5) ($t_l < t_2$): Recall that $C^* = 0$ on $[\tilde{t}_1, T]$. The C^* policy is one of the types given by

(i) $\quad C^* = 0 \quad$ for $\quad t \in [0, \tilde{t}_1)$,

(ii) $\quad C^* = 0 \quad$ for $\quad t \in [0, t_1')$,

$$C^* = C^M \quad \text{for} \quad t \in [t_1', t_1''), \tag{35}$$

$\quad C^* = 0 \quad$ for $\quad t \in [t_1'', \tilde{t}_1)$,

(iii) $\quad C^* = C^M \quad$ for $\quad t \in [0, \hat{t}_1)$,

$\quad C^* = 0 \quad$ for $\quad t \in [\hat{t}_1, \tilde{t}_1)$.

We conclude the analysis of the shareholders' problem with some economic interpretations of the optimal selling policies, as well as the optimal dividend policy. For the selling policies, it suffices to interpret a selling policy of type C. Using (26)–(27) yields the result that P_1 should sell shares at the maximal rate for all t, if and only if

$$\left[(1-\tau_g)R^*(T)+\bar{Z}\right]\Big/\bar{Z}+\int_0^T\left[(1-\tau_1)C^*(t)+v^1(t)\right]\mathrm{d}t+\gamma^1<p. \tag{36}$$

This amounts to saying that P_1 should sell at the maximal rate if the marginal value, at the initial instant, of keeping a share is less than what could be obtained by selling the unit. (Since $\lambda_1^1(t)<\lambda_1^1(0)$, $\forall t\in(0,T]$, the argument applies equally well for all $t\in(0,T]$). Inequality (36) can also be written as

$$\left[(1-\tau_g)R^*(T)+\bar{Z}\right]\frac{Z_{10}}{\bar{Z}}+\int_0^T(1-\tau_1)C^*(t)Z_{10}\,\mathrm{d}t+\int_0^T v^1(t)Z_{10}\,\mathrm{d}t$$

$$<Z_{10}(p-\gamma^1). \tag{37}$$

Recall that $Z_{10}>\bar{Z}/2$. On the left-hand side of (37) the first term is the capital gain to be collected at $t=T$ if P_1 does not sell shares. The second term is the accumulated dividend in the case of no selling. The last term is nonnegative and identically zero if the constraint $Z_1\geq\bar{Z}/2$ never binds. The term on the right-hand side is the (adjusted) sales value of Z_{10}. Hence, if (37) holds, P_1 will be better off by selling all his initial stock of shares since the sales value exceeds what can be collected in capital gain and dividends. But notice that he cannot sell his initial amount of shares instantaneously; the best to do is to decrease Z_{10} as fast as possible by selling at the maximal rate.

The expression (34) can be economically interpreted in the following way. At any instant the firm has the possibility of using a dollar of its cash flow to pay out as dividend or, alternatively, to retain the dollar and

- pay back a dollar of debt (or, if debt is already negative, to lend one dollar more),
- finance a dollar of investment.

Notice that payment of interest on debt and corporate tax, i.e., rY and $f[G(K)-aK-rY]$, is mandatory and leaves no choice to the firm. To shareholder P_1 the investment policy is given; hence P_1 can only choose between dividends and/or reducing debt/increasing lending. In (34), the term $(1-\tau_1)Z_1/\bar{Z}$ represents the net amount which P_1 receives if one dollar of dividend is paid out at time t. Recall that λ_3^1 has an interpretation as the marginal contribution to optimal profits, caused by a marginal increase in retained earnings (R). Hence, as long as the net benefit from one present

Table 3

Qualitative conditions for occurrence of various dividend policies.

C^* policy type	Tax rates	Initial amount of shares	Net cost of debt
1	$\tau_1 \gg \tau_g$	$Z_{10} \cong \bar{Z}/2$	$(1-f)r$ large
2	$\tau_1 \cong \tau_g$	$Z_{10} \gg \bar{Z}/2$	
3	$\tau_1 \cong \tau_g$	$Z_{10} \gg \bar{Z}/2$	$(1-f)r$ small

dollar of dividend exceeds the value of retaining the dollar, dividends should be paid out, and vice versa. The second expression in (34) can also be interpreted economically. The right-hand side represents the (net of capital gains tax) amount which P_1 collects at $t = T$ if the dollar at hand is used for decreasing the debt. If debt is decreased by one dollar, then the instantaneous interest cost is reduced by $(1-f)r$; the value of this saving over the interval $[t,T]$ equals

$$\int_t^T (1-f)r\exp\{(1-f)rs\}\,\mathrm{d}s = \exp\{(1-f)r(T-t)\}.$$

Hence the term $(1-\tau_g)Z_1\exp\{(1-f)r(T-t)\}/\bar{Z}$ represents P_1's share of the interest cost saved by not paying out a dollar of dividend at time t. (If the firm lends money, an additional dollar yields interest income in amount of $\exp\{(1-f)r(T-t)\}$ on $[t,T]$, and similar arguments as for the debt case apply.)

It is easy to establish some qualitative conditions for the occurrence of the three dividend policies in (35). In table 3 such conditions are stated. Under policy 1, no dividends are paid out since the decision is made by the majority shareholder (P_1) who suffers from a high personal tax rate on dividends (τ_1), has only a small majority, and the net cost of debt is large. In view of his objective, dividends are discouraged by the high value of τ_1 and the relatively small amount of shares in possession (Z_1). The net cost of debt being large implies that a high value of $R(T)$ (which is desirable) should be achieved by a cautious dividend policy rather than expanding K through debt-financed investments. Under policy 3, dividends are initially paid out, motivated by a relatively small personal tax rate of P_1, a comfortable amount of shares (which increases the total amount of dividends received, CZ_1), and a net cost of debt being small. Here, a certain amount of dividends can be defended since taxation on dividends now, and retained earnings later, is approximately the same. Moreover, the loss of retained earnings incurred by the dividend payout can be counterbalanced by attracting debt money (to invest and increase K) since the cost of such funds is relatively low.

3.2. The manager's problem

The optimization problem of the manager consists of selecting a piecewise continuous investment rate $I(t)$, such that $0 \le I(t) \le I^M$, to maximize the payoff functional J given by (10), subject to the original state equations (4), (5), (7), the six costate equations of the followers with their appropriate boundary conditions, and the state variable inequality constraint (15).

Let $\mu(t)$, $\eta_1(t)$, and $\eta_2(t)$ be piecewise continuous multiplier functions and let $\lambda_1(t), \lambda_2(t), \dots, \lambda_9(t)$ be piecewise continuously differentiable costate variables. Let λ_0 be a nonnegative constant.

It may be convenient to transform the payoff (10). Using (4) we obtain

$$J = \int_0^T (1-f)[G(K) - aK - r(K - \overline{Z} - R)]\,\mathrm{d}t$$

$$= \int_0^T \dot{R}\,\mathrm{d}t + \overline{Z}\int_0^T C^*\,\mathrm{d}t$$

$$= \int_0^T C^*\overline{Z}\,\mathrm{d}t + R(T) - R_0.$$

The Hamiltonian becomes

$$H = \lambda_0 C^*\overline{Z} + \lambda_1(S_2^* - S_1^*) + \lambda_2(I - aK)$$

$$+\lambda_3[(1-f)[G(K) - aK - r(K - \overline{Z} - R)] - C^*\overline{Z}]$$

$$+\lambda_4(-(1-\tau_1)C^* - v^1) + \lambda_5[a\lambda_2^1 - \lambda_3^1(1-f)(G'(K) - a - r)]$$

$$+\lambda_6(-\lambda_3^1(1-f)r) + \lambda_7((1-\tau_2)C^* + v^2)$$

$$+\lambda_8[a\lambda_2^2 - \lambda_3^2(1-f)(G'(K) - a - r)] + \lambda_9(-\lambda_3^2(1-f)r),$$

and the Lagrangian is given by

$$L = H + \mu[(1+k)(R + \overline{Z}) - K] + \eta_1 I + \eta_2(I^M - I).$$

The set of necessary conditions is as follows:

$$I^* = \underset{0 \le I \le I^M}{\arg\max}\, H, \tag{38}$$

$$\frac{\partial L}{\partial I} = \lambda_2 + \eta_1 - \eta_2 = 0, \tag{39}$$

$$\dot{\lambda}_1 = \frac{\mathrm{d}C^*}{\mathrm{d}Z_1}\left[\bar{Z}(\lambda_3 - \lambda_0) + \lambda_4(1-\tau_1) - \lambda_7(1-\tau_2)\right], \tag{40a}$$

$$\begin{aligned}\dot{\lambda}_2 = {} & a\lambda_2 - (1-f)(G'(K) - a - r)\lambda_3 \\ & + (1-f)G''(K)(\lambda_5\lambda_3^1 + \lambda_8\lambda_3^2) + \mu,\end{aligned} \tag{40b}$$

$$\dot{\lambda}_3 = -(1-f)r\lambda_3 - \mu(1+k), \tag{40c}$$

$$\dot{\lambda}_4 = \lambda_1 \frac{\mathrm{d}S_1^*}{\mathrm{d}\lambda_1^1}, \tag{40d}$$

$$\dot{\lambda}_5 = -a\lambda_5, \tag{40e}$$

$$\dot{\lambda}_6 = (1-f)r\lambda_6 + \lambda_5(1-f)(G'(K) - a - r), \tag{40f}$$

$$\dot{\lambda}_7 = -\lambda_1 \frac{\mathrm{d}S_2^*}{\mathrm{d}\lambda_1^2}, \tag{40g}$$

$$\dot{\lambda}_8 = -a\lambda_8, \tag{40h}$$

$$\dot{\lambda}_9 = (1-f)r\lambda_9 + \lambda_8(1-f)(G'(K) - a - r), \tag{40i}$$

$$\mu \ge 0, \quad \mu\left[(1+k)(R+\bar{Z}) - K\right] = 0, \tag{41a}$$

$$\eta_1 I = 0, \quad \eta_1 \ge 0, \quad \eta_2(I^M - I) = 0, \quad \eta_2 \ge 0, \tag{41b}$$

$$\lambda_1(T) = \left[\lambda_9(T) - \lambda_6(T)\right](1-\tau_g)/\bar{Z}, \tag{42a}$$

$$\lambda_2(T) = -\alpha, \tag{42b}$$

$$\lambda_3(T) = \alpha(1+k) + \left[\lambda_7(T) - \lambda_4(T)\right](1-\tau_g)/\bar{Z} + \lambda_0, \tag{42c}$$

$$\alpha \ge 0, \quad \alpha\left[(1+k)(R(T)+\bar{Z}) - K(T)\right] = 0, \tag{43}$$

$$\lambda_4(0) = \lambda_5(0) = \lambda_6(0) = \lambda_7(0) = \lambda_8(0) = \lambda_9(0) = 0. \tag{44}$$

From (40e), (40f), (40h), (40i), and (44), it appears that

$$\lambda_5 = \lambda_6 = \lambda_8 = \lambda_9 = 0 \quad \text{for} \quad 0 \leq t \leq T, \tag{45}$$

which is intuitively reasonable since the followers' costates $\lambda_2^1, \lambda_3^1, \lambda_2^2, \lambda_3^2$ do not have direct significance for the manager's problem.

We shall make two assumptions that will simplify the analysis of the optimality conditions. First, assume that I^M is sufficiently large such that a singular I is always feasible, that is, the constraint $I \leq I^M$ will never bind. (This assumption is made for mathematical convenience but seems reasonable in the light of the model's financial structure. The model *per se* imposes unlimited investments, neither by borrowing money nor by retaining profits.) The assumption implies that the multiplier η_2 is identically zero. Set $\eta_1 = \eta$. Second, to avoid contraction policies, assume that

$$G'(K) > a + r \quad \text{for} \quad K = K_0. \tag{46}$$

Let us start the analysis with some observations concerning an optimal, positive investment rate. (Obviously, for $\lambda_2 < 0$, no investment occurs). From the Hamiltonian we see that I will be singular whenever $\lambda_2 = 0$. A necessary condition for optimality of a singular control is given by the generalized Legendre–Clebsch condition which requires that

$$(1-f)\lambda_3 G''(K) \leq 0,$$

which holds only if $\lambda_3 \geq 0$. Notice that the costate λ_2 is the shadow price of a unit of K, as assessed by the manager, and λ_3 is the shadow price of a unit of R. Obviously, as seen above, $\lambda_2 < 0$ implies zero investment and, moreover, $\lambda_3 < 0$ implies that singular, positive investment cannot be optimal. On a singular path it holds that $\dot{\lambda}_2 = \ddot{\lambda}_2 = 0$. Hence,

$$\dot{\lambda}_2 = -\lambda_3(1-f)(G'(K) - a - r) + \mu = 0, \tag{47}$$

and, whenever μ is differentiable,

$$\ddot{\lambda}_2 = \dot{\mu} - (1-f)\dot{\lambda}_3(G'(K) - a - r) - \lambda_3(1-f)G''(K)(I - aK)$$
$$= 0. \tag{48}$$

Substituting from (5) and (40c) into (48) yields

$$I^s = aK + \frac{\dot{\mu}}{\lambda_3(1-f)G''(K)}$$
$$+ \frac{\mu}{\lambda_3(1-f)G''(K)}\left[(1-f)r\lambda_3 + \mu(1+k)\right], \tag{49}$$

which shows that I^s equals aK (i.e., investment is just at the replacement level) if the state constraint $(1+k)(\bar{Z}+R) \geq K$ does not bind. For $I^s = aK$, the corresponding value of K, say, K^s, is implicitly given as the (unique) solution of

$$G'(K^s) = a + r, \tag{50}$$

which follows from (47).[8] Hence, $I^s = aK^s$ is constant.

Next we characterize four possible paths by using the complementary slackness conditions.

Path	1	2	3	4
μ	+	0	0	+
η	0	0	+	+

Path 1: This is a boundary path – $K = (1+k)(\bar{Z}+R)$. Moreover, $\lambda_2 = 0$ and the control I is singular – I^s is given by (49). The control, say, I^b, which will maintain K equal to $(1+k)(\bar{Z}+R)$ is given by

$$I^b = aK + (1+k)\big[(1-f) \times \big[G(K) + aK - r(K - \bar{Z} - R)\big] - C^*\bar{Z}\big]. \tag{51}$$

From $\dot{\lambda}_2 = \lambda_2 = 0$, we obtain

$$\mu = (1-f)(G'(K) - a - r)\lambda_3 > 0 \Rightarrow G'(K) > a + r, \tag{52}$$

whenever $\lambda_3 > 0$. (Notice that, if $\lambda_3 = 0$, then path 1 cannot occur.) Next, observe that

$$I^b \gtreqless aK \leftrightarrow \dot{K}, \dot{R} \gtreqless 0.$$

It is easy to show that, if $C^* = 0$ for all t on path 1, then $\dot{R} > 0$. However, $\dot{R} > 0$ may not hold in general, but $\dot{R} > 0$ on a final interval since $C^* = 0$ in such an interval. Path 1 is a feasible final path since the transversality condition $\lambda_2(T) = 0$ is satisfied. However, if $\dot{K} > 0$, then $G'(K)$ decreases and, if path 1 is extended on a sufficiently long interval, it may happen that $G'(K) > a + r$ is violated.

[8]Strictly speaking $G'(K) = a + r$ does not need to hold in (47) if $\lambda_3 = 0$.

Path 2. On this path, where $\mu = \eta = 0$, we have $(1+k)(\bar{Z}+R) \geq K$ and $I \geq 0$. Then (49) yields $I^s = aK^s$ and K^s is given by (50) whenever $\lambda_3 > 0$.[9] To satisfy (39), $\lambda_2 = 0$ must hold which makes path 2 a feasible final path. Notice that $\dot{K} = 0$ and hence $\dot{Y} = \dot{R} = 0$ or $\operatorname{sgn}(\dot{Y}) = -\operatorname{sgn}(\dot{R})$.

Path 3. On this path we have $\mu = 0$ and $(1+k)(\bar{Z}+R) \geq K$. Moreover, $\eta > 0$ implies $I = 0$ and hence $\dot{K} < 0$. It must hold that at least one of $\dot{R}, \dot{Y}$ is negative. Notice that path 3 is infeasible as a final path since $\lambda_2 = -\eta < 0$.

Path 4. This is a boundary path and $I = 0$. Furthermore, $\dot{K}, \dot{R}, \dot{Y} < 0$ and $\lambda_2 = -\eta < 0$ makes path 4 infeasible as a final path. Notice that $\dot{R}$ decreases irrespective of whether $C = 0$ or $C > 0$.

An initial feasible path is a path which satisfies the fixed initial conditions. If we assume that the firm has maximal debt at $t = 0$, the initial values $K(0)$, $R(0)$ must satisfy[10]

$$(1+k)(\bar{Z}+R(0)) = K(0). \tag{53}$$

Van Loon (1983) argues that, if we in (53) had strict inequality (i.e., debt less than maximal), the firm would instantaneously attract the missing amount of debt and invest it. After that, the firm start on a feasible path. A mathematically stringent argumentation for assumption (53) can be found in Feichtinger and Hartl (1986, p. 378).

Recall that paths 3 and 4 are infeasible as final paths; path 2 is a feasible final path and path 1 may be a feasible path. The procedure is now to work backwards from $t = T$ and consider a feasible final path. A first question to answer is the following: is path 1 or path 2 a candidate for an optimal solution for the entire planning period? Path 1 is a candidate for an optimal solution for all t only if $G'(K) > a + r$ holds throughout the interval $[0, T]$. It is, however, questionable if this actually will be satisfied. Path 2, on the other hand, can never be optimal on $[0, T]$ if $\lambda_3(0) > 0$ since (46) is then violated.

The next step is to determine which path can precede a final path. Therefore we test for each feasible final path which paths can precede those final paths. The testing procedure utilizes the properties of paths 1–4 described above as well as continuity properties of state and costate variables. If the set of feasible preceding paths is not empty we repeat the coupling procedure. The procedure stops when no more paths can precede a feasible string of policies. Note that

[9] If $\lambda_3 = 0$ on an interval, then K^s is not uniquely determined by (47), that is, $G'(K) = a + r$ does not necessarily hold. This could cause difficulties in the coupling procedure to follow since the arguments employed require that $G'(K) = a + r$ be satisfied. In appendix 4 this issue is discussed in more detail.

[10] See van Loon (1983), van Schijndel (1987).

the initial condition (53) must hold. [A precise description of the coupling technique can be found in appendix 5; see also van Loon (1983).] It turns out that there is only one coupling satisfying all necessary conditions. We can prove the following proposition:

Proposition 1. The only policy strings consisting of paths 1–4 that satisfy the necessary optimality conditions are

(*) Path 1 → Path 2,

(* *) Path 1 throughout $[0, T]$.

Proof. See appendix 5.

In figs. 1 and 2 we have depicted the evolution of some key variables for the case of C policies of type 1 and 3 (cf. table 3) for the case of a policy string 'path 1 → path 2'. In fig. 1 we have the case of *zero dividends*. It occurs if, among other things, the majority shareholder's personal tax rate is high and/or the net cost of debt is large. Initially (on path 1) K, R, and Y are all increasing and debt is maximal. Gross investment, I, is greater than depreciation, aK, which implies an increasing stock of capital goods. Retained earnings, R, increase since no dividends are paid out and $G'(K) > a + r$ on path 1. The latter condition means that a marginal unit of investment gives the firm a return, $G'(K) - a$, being greater than the interest rate. This justifies that debt is increased (maximally). However, since K increases, $G'(K)$ decreases, and at $t = t_{12}$, $G'(K)$ reaches its stationary level where $G'(K) = a + r$, and we get on path 2. Here, K is constant and R is increasing. Investment is at the replacement level and since retained earnings still increase, the remaining cash is used to pay off debt, i.e., Y decreases. Depending on the parameters of the problem, lending may occur from some instant, say, t_0. On path 2 the evolution of Y is given by

$$\dot{Y} = -(1-f)(G(K^s) - aK^2 - rY),$$

which has the solution

$$Y = \frac{G(K^s) - aK^s}{r} + \left[Y_{12} - \frac{G(K^s) - aK^s}{r}\right] \exp\{-(1-f)r(t_{12} - t)\},$$

where Y_{12} is the level of debt at the coupling instant $t = t_{12}$. In fig. 1 the

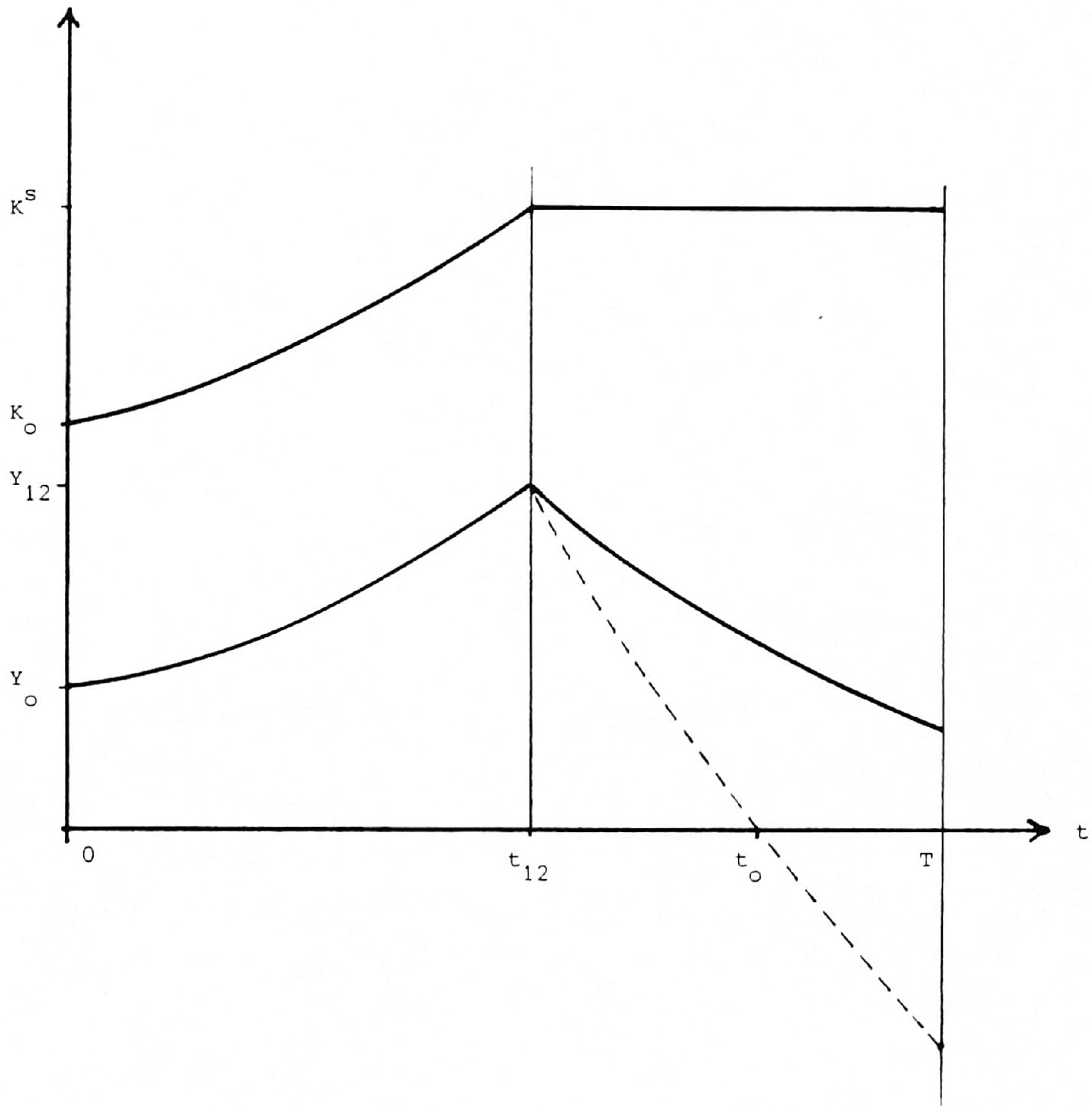


Fig. 1. Optimal policies for capital stock (K) and debt/lending (Y) when no dividends are paid out.

instant t_0 (where lending starts) is given by

$$t_0 = t_{12} - \frac{1}{(1-f)r} \ln\left(G(K^s) - aK^s - rY_{12}\right) \Big/ \left(G(K^s) - aK^s\right). \tag{54}$$

Notice that, if $t_0 > T$, then $Y > 0$ for all $t \in [0, T]$ and lending does not occur. Eq. (54) shows that a no-lending case emerges if t_{12} and Y_{12} are large, i.e., the expansion period is long compared to the period of stationary evolution, and the level of debt incurred after the expansion period is large. This seems to be intuitively reasonable.

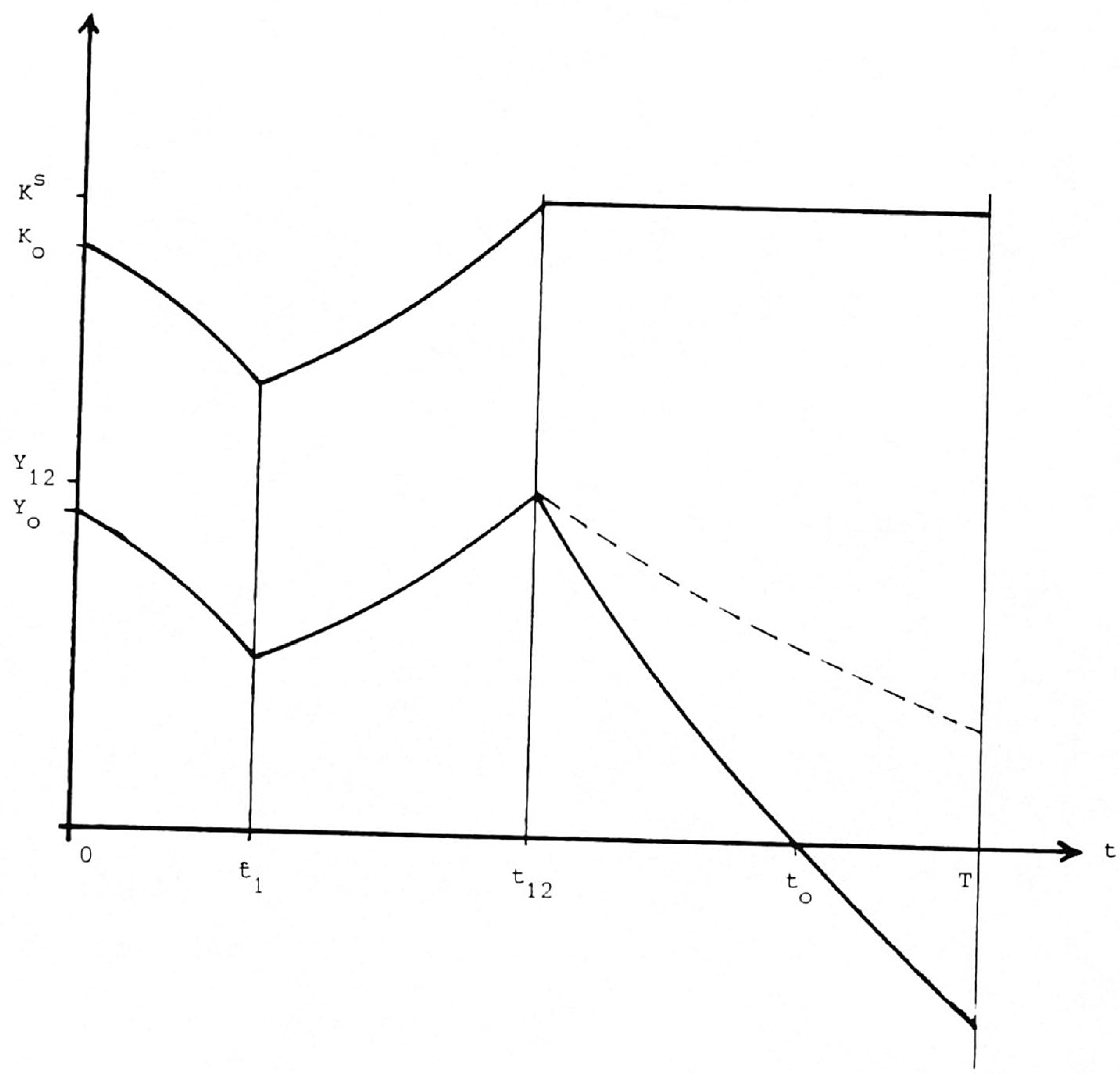


Fig. 2. Optimal policies for capital stock (K) and debt/lending (Y) when maximal dividends are paid out on an initial interval ending at $t = \hat{t}_1$ such that $\hat{t}_1 \leq t_{12}$.

In Lemma 5 we prove that, if the interest rate r is sufficiently large (which implies that the net cost of debt, $(1-f)r$, is large), lending is unlikely to occur. Notice that we still consider the case of zero dividends.

Lemma 5. The instant t_0 given by (54) is increasing as a function of the parameter r, i.e., $\mathrm{d}t_0/\mathrm{d}r$ is positive.

Proof. See appendix 6.

The lemma states that for increasing values of r, t_0 increases and $t_0 > T$ is likely to be the case, implying that Y is positive for all $t \in [0, T]$. Hence, if debt is very costly, no dividends will be paid out (cf. table 3) and, moreover, lending is unlikely to occur.

In fig. 2 we turn to the case of *maximal dividend payment on an initial interval, followed by zero dividends for the rest of the planning period*. This case occurs if, for example, the majority shareholder's personal tax rate is comfortably low, see also table 3. Consider fig. 2 where the dividend policy switches *before* the coupling instant t_{12}. On the intervals $[\hat{t}_1, t_{12})$ and $[t_{12}, T]$ the evolution of K, R, and Y are the same (qualitatively speaking) as in fig. 1. Notice that, if r is very low, lending (on the interval $(t_0, T]$) could occur here. Depending on the actual value of C^M, two situations can be distinguished.

(a) If C^M is sufficiently low, $\dot{R} = (1-f)(G(K) - aK - rY) - C^M\overline{Z}$ remains positive on $[0, t_{12}]$ and fig. 1 applies.

(b) For C^M sufficiently large, $\dot{R}$ becomes negative and K as well as Y decrease on the interval $[0, \hat{t}_1)$. See fig. 2. In such a situation the firm initially follows a contraction policy. Investment is below the replacement level, implying a decreasing K. Even if investment is low, the cumulative retained earnings decrease since large amounts of dividends are distributed and debt is paid off at the same time.

If the dividend policy switches *after* the coupling instant, i.e., $\hat{t}_1 > t_{12}$, the value of C^M again becomes significant.

(α) For C^M sufficiently low, the situation will be as in fig. 1.

(β) For large values of C^M, $\dot{R}$ becomes negative, implying that K decreases. But $\dot{K} > 0$ must hold on some interval before the coupling instant t_{12}; cf. appendix 5. Hence, for $\hat{t}_1 > t_{12}$ and in case of a large value of C^M, the feasibility of the string path 1 → part 2 may be lost.

To conclude this section, we briefly study the case $T \to 0$, that is, the case of a 'small' planning period. This can be done in two ways. First, we could see what happens to the solution when, formally, T tends to zero. Thus, we study the limiting game $T \to 0$. For practical purposes this amounts to analyzing the game with a small, but finite T. Second, a static game could be played at $t = 0$. Such a game, however, does not make very much sense in the scenario at hand, and it turns out that its solution is rather trivial. Hence we consider the first alternative.

For a sufficiently short planning horizon, there is reason to believe that the optimal selling policies of the shareholders will be $S_1^* = S_2^* = S^M$ [cf. Case (9) in table 2]. From table 1 we see that, for T close to zero, the condition for occurrence of the maximum-selling policy (approximately) becomes

$$\left[(1-\tau_g)R(T) + \overline{Z}\right]\frac{Z_{i0}}{\overline{Z}} < pZ_{i0}, \tag{55}$$

assuming that the state constraints $Z_1(T) \geq \bar{Z}/2$ and $Z_2(T) \geq 0$, respectively, are not binding. (This is likely to be the case when T is small). Hence, maximal selling is optimal if the capital gain to be collected at $t = T$, if no shares are sold on $[0, T]$, does not exceed the proceeds to be obtained by selling the whole initial amount of shares. Eq. (3) shows that, for a small T, $R(T) \cong R_0$, and (55) will hold if R_0 is not too large. In summary, if the firm starts with a relatively small amount of retained earnings and the planning horizon is small, we expect the shareholders to sell their initial holdings of shares at the maximal rate. This is due to the fact that retained earnings cannot increase that much as to make the collection of capital gains better than getting the proceeds from selling out the initial volume of shares. In this case we obtain $C^* \equiv 0$ which, in a sense, confirms the use of maximal selling.

For the manager the solution of fig. 2 applies, at least qualitatively. [Notice that in the present case, an initial phase of contraction (as in fig. 2) cannot occur.] Due to the complexity of the manager's problem and the fact that the functions involved are partially unspecified, it seems difficult to characterize precisely what happens to the solution of the manager's problem for $T \to 0$. But from the payoff functional of the manager (cf. section 3.2) it can be seen that $J \cong 0$ for $C^* \equiv 0$ and $R(T) \cong R_0$. Hence, whatever paths for investment and debt the manager might choose, his payoff will be negligible. We may conclude that in the case of a small T (and a low initial amount of retained earnings) the manager's problem tends to lose its significance.

4. Concluding remarks

In this paper we studied a problem in the areas of 'the dynamics of the firm' and 'corporate finance'. A deterministic, dynamic model was set up with the purpose of characterizing optimal investment, financing, and dividend policies of a firm with separation of management and ownership. In the latter respect, the present work differs from, for instance, van Loon (1983) and van Schijndel (1987) where no such separation exists. To model the possible conflicts between management and shareholders, a Stackelberg-differential game approach was applied and with a view to tractability we assumed open-loop controls.

More specifically, within the framework of a financial model of the firm, we assumed that a manager controls the firm's investment policy over a fixed planning period. With the manager being the Stackelberg leader, the shareholders respond rationally to the announced investment policy by choosing a dividend policy and policies for the internal trade of shares. The dividend policy is decided by the majority shareholder and each shareholder receives dividends in proportion to the fraction of shares he possesses. At the end of the game the owners receive their respective parts of the corporate assets.

An important aspect of the scenario is the presence of taxation. Here, we considered corporate as well as personal taxes; the latter being charged on the streams of income and on the terminal capital gains.

The solution of the Nash game played by the shareholders was obtained by standard methods of optimal control. Due to linearity, the dividend policy as well as the share trading policies turned out to be bang-bang policies. The manager's problem was only solvable in a qualitative way and we applied a path-connecting procedure designed by van Loon (1983). Here, the optimal string of paths was a simple two-path sequence. At a terminal interval the investment policy is designed to maintain the stock of capital goods at an optimal stationary level and debt is gradually paid off. Depending on the parameters of the problem even lending may occur during this final phase. The initial phase is an expansion phase if no dividends are paid out; for a sufficiently large rate of dividend pay out, an initial time interval of contraction can occur, however.

In order to obtain our results, a number of assumptions were made. Those we find most crucial are the following:

(1) The firm is in some respects 'a closed system'. Although debt money can be attracted/paid off and lending is possible, the amount of common stock is fixed. This means that funds cannot be obtained by emissions of new shares. Moreover, the existing shareholders were not permitted to buy/sell shares from/to investors outside the firm.[11]

(2) The price of a share traded between the shareholders was considered fixed and constant. This assumption obviously deprives the shareholders from a range of interesting options.

(3) Control of the dividend policy cannot switch during the play. This means that the shareholder who initially has the majority of shares will continue to be in this position throughout the planning period. Here we assumed that the high-taxed shareholder had the majority of shares. Of course, different results would have been obtained if the low-taxed shareholder has had the majority.

(4) The strategies of the manager as well as the shareholders are open-loop, implying that the players are supposed to stick to predetermined plans that are independent of the current state of the game.

An obvious task for future research would be to relax these assumptions. However, one should be prepared to face considerable difficulties in the set-up as well as the analysis of such a model.

[11] The assumption of no new share emissions may not be unrealistic. Kirg and Fullerton (1984) state that in their U.S. study, new shares issues accounted for only 8% of all new equity finance for (nonfinancial) corporations over the period 1970–79 and the tax advantage of internal finance over new shares issues appeared to be quite large.

Appendix 1: Proof of Lemma 2

We shall prove that the regimes (2), (3), (4), and (7) in table 2 are infeasible. Consider *regime (3)*: It occurs if

$$\kappa + \gamma^1 > p, \tag{A.1}$$

and

$$\kappa + \gamma^2 + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]\, \mathrm{d}s < p. \tag{A.2}$$

If $\gamma^2 > 0$, then $\gamma^1 = 0$ and the inequalities (A.1)–(A.2) cannot be satisfied. [This is also true for $\gamma^1 = \gamma^2 = 0$.] We conclude that regime (3) occurs only if $\gamma^1 > 0$. This implies $Z_1(T) = \overline{Z}/2$,

$$\kappa + \gamma^1 > p,$$

and

$$\kappa + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]\, \mathrm{d}s < p.$$

By a similar reasoning, *regime (7)* occurs only if $\gamma^2 > 0$ implying $Z_1(T) = \overline{Z}$,

$$\kappa + \gamma^2 > p,$$

and

$$\kappa + \int_0^T [(1-\tau_1)C^*(s) + v^1(s)]\, \mathrm{d}s < p.$$

Consider *regime (2)* which occurs if

$$\kappa + \gamma^1 > p$$

and

$$\{\kappa + \gamma^2 < p\} \wedge \left\{\kappa + \gamma^2 + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]\, \mathrm{d}s > p\right\}.$$

This case occurs only if $\gamma^1 > 0$,

$$\kappa + \gamma^1 > p,$$

and

$$\{\kappa < p\} \wedge \left\{\kappa + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]\, \mathrm{d}s > p\right\}.$$

Similarly, *regime (4)* occurs only if $\gamma^2 > 0$,

$$\kappa + \gamma^2 > p,$$

and

$$\{\kappa < p\} \wedge \left\{\kappa + \int_0^T [(1-\tau_1)C^*(s) + v^1(s)]\, \mathrm{d}s > p\right\}.$$

[Note that regimes (2) and (3) occur only if $\gamma^1 > 0$, implying $Z_1(T) = \bar{Z}/2$. This makes some economic sense since regimes (2)–(3) both have $S_1 \equiv 0$. Hence, knowing that his majority could ultimately be lost, P_1 prefers not to sell at all. A similar interpretation applies to regimes (4) and (7) where $\gamma^2 > 0$, implying $Z_1(T) = \bar{Z}$, i.e., $Z_2(T) = 0$.]

For regimes (2) and (3) we have $\dot{Z}_1 \geq 0$, implying $Z_1(T) > Z_{10} > \bar{Z}/2$ (implying $\gamma^1 = 0$) which contradicts $Z_1(T) = \bar{Z}/2$ (being implied by $\gamma^1 > 0$). Hence these two regimes are infeasible.

For regimes (4) and (7) we have $\dot{Z}_1 \leq 0$, implying $Z_1(T) < \bar{Z}$ (implying $\gamma^2 = 0$) which contradicts $Z_1(T) = \bar{Z}$ (being implied by $\gamma^2 > 0$). Hence these regimes are infeasible too. Q.E.D.

Appendix 2: Proof of Lemma 3

First notice that having $t_1 > t_2$ in regime (5) implies that $\dot{Z}_1 \geq 0$ for all t. When Z_1 is nondecreasing, C^* is identically equal to zero [cf. (32)]. Moreover, the constraint (11) cannot become binding; hence $v^1 = 0$ for all t, and $\gamma^1 = 0$. Using (21a) shows that $\dot{\lambda}_1^1 = 0$ for all t, implying that $\lambda_1^1 \equiv \kappa$. Using (26)–(27) and table 1 we observe that S_1^* cannot switch and table 1 shows that, if $\kappa > p$, $S_1^* \equiv 0$, and we conclude that $S_2^* \equiv 0$, i.e., $\dot{Z}_1 \equiv 0$. In summary, we have $t_1 = t_2 = T$ as in regime (1) if $\kappa < p$, $S_1^* \equiv S^M$, and S_2^* must be identically equal to S^M in order to have $\dot{Z}_1 \geq 0$. Hence $\dot{Z}_1 \equiv 0$ and $t_1 = t_2 = 0$ as in regime (9). Q.E.D.

Appendix 3: Proof of Lemma 4

For a singular C it must hold that

$$Z_1 = \lambda_3^1 \frac{\bar{Z}}{1-\tau_1} \quad \text{and} \quad \dot{Z}_1 = \dot{\lambda}_3^1 \frac{\bar{Z}}{1-\tau_1}.$$

Using (21c) yields

$$\dot{Z}_1 = -\lambda_3^1(1-f)r\frac{\bar{Z}}{1-\tau_1} = -(1-f)rZ_1,$$

which implies, by (7), that

$$S_2 - S_1 = -(1-f)rZ_1.$$

Since for $C > 0$ the S_i's are strictly bang-bang and $Z_1 > 0$, it must hold that $S_2 = 0$ and $S_1 = S^M$. Thus,

$$S^M = (1-f)rZ_1 \Rightarrow Z_1 = S^M/((1-f)r),$$

which yields $\dot{Z}_1 = 0$ and $S_1 = S_2$. But this contradicts $S_1 = S^M$, $S_2 = 0$, and we conclude that a singular C is not feasible. Q.E.D.

Appendix 4

In this appendix we deal with the expression (47) which uniquely determines K^s and I^s on path 2 iff $\lambda_3 > 0$ throughout this path. In relation to the coupling procedure described in appendix 5 the crucial question is the following. What difference in the results of appendix 5 does it make if, roughly speaking, $\lambda_3 = 0$ at the instant where path 2 is coupled (before or after) another path? From appendix 5 it appears that the following cases must be dealt with:

(A) Path 2 → Path 1.
(B) Path 4 → Path 2; Path 3 → Path 2; Path 1 → Path 2.

(A): First notice that on path 2 we have $\lambda_3(G'(K) - a - r) = 0$ from (47). Let t_{21}^- denote an instant just before the coupling instant t_{21} and let $\lambda_3(t_{21}^-) = 0$. Hence $G'(K) - a - r$ is undetermined at $t = t_{21}^-$ and we may have

$$(1)\quad G' < a + r \quad \text{or} \quad (2)\quad G' = a + r \quad \text{or} \quad (3)\quad G' > a + r.$$

If (1) holds, path 2 cannot be coupled before path 1 since this would require a jump in K. Recall that $G' > a + r$ on path 1.

If (2) holds then the arguments of appendix 7 (path 2 → path 1 → path 2) show that coupling is impossible. If (3) holds, consider the costate equation (40c) just before and just after the coupling instant t_{21}. Assume that t_{21} is not a switching point of the dividend policy C. On path 2 we have

$$\dot{\lambda}_3(t_{21}^-) = \lambda_3(t_{21}^-) = 0,$$

whereas on path 1 (t_{21}^+ denoting 'just after' t_{21})

$$\dot{\lambda}_3(t_{21}^+) = -(1-f)r\lambda_3(t_{21}^+) - \mu(1+k).$$

Whenever λ_3 is continuous at $t = t_{21}$, $\lambda_3(t_{21}^-) = \lambda_3(t_{21}^+)$, implying $\dot{\lambda}_3(t_{21}^+) = -\mu(1+k) < 0$ and $\lambda_3(t_{21}^+) < 0$. This, however, violates the necessary Legendre–Clebsch condition that λ_3 be non-negative. Hence, if $\lambda_3 = 0$ at $t = t_{21}^-$, path 2 cannot be coupled before path 1. Following Feichtinger and Hartl (1986, corol. 6.3), we know that λ_3 is continuous if:

(i) I is continuous at t_{21} and the following constraint qualification (CQ) is satisfied:

$$\text{vectors } (1, I, 0) \quad \text{and} \quad (-1, 0, (1+k)(R+\bar{Z}) - K)$$

be linearly independent. CQ is satisfied for $I > 0$, but not for $I = 0$. The latter case is dealt with below.

(ii) I is discontinuous at t_{21} and t_{21} is an entry point where entry is nontangential. Obviously, entry will be nontangential.

If $I(t_{21}) = 0$, then CQ is not satisfied and continuity of λ_3 is not guaranteed. For this case we apply the following argument to prove infeasibility of the coupling path 2 → path 1. We need to distinguish the cases $C(t_{21}) = 0$ and $C(t_{21}) = C^M$:

(a) $C(t_{21}) = 0$. From (59) we have

$$0 = aK + (1+k)(1-f)\left[G(K) - \left(a + \frac{rk}{1+k}\right)K\right]. \tag{A.3}$$

But $G'(K) > a + r \Rightarrow G(K) > G'(K)K > (a+r)K > (a + rk/(1+k))K$, which shows that (A.3) cannot be satisfied. Hence, with $I(t_{21}) = 0$ the coupling is infeasible.

(b) $C(t_{21}) = C^M$. From (51) we have

$$C^M\bar{Z} = aK + (1+k)(1-f)\left[G(K) - \left(a + \frac{rk}{1+k}\right)K\right]. \tag{A.4}$$

We choose to regard (A.4) as a borderline case; it would only be by coincidence that $K(t_{21})$ would satisfy (A.4). Hence, with $I(t_{21})=0$ coupling is infeasible.

(**B**): *Path 2 as final path.* From (47c) we obtain that on path 2

$$\lambda_3=\lambda_3(T)e^{(1-f)r(T-t)}.$$

Recall that $\lambda_3 \geq 0$ is necessary for a singular path. Hence $\lambda_3 \geq 0 \Leftarrow \lambda_3(T) \geq 0$. Eq. (40c) shows that $\dot{\lambda}_3 \leq 0$ for $\lambda_3 \geq 0$. If $\lambda_3(T)>0$, then, at a coupling instant t_{j2} $(j=1,3,4)$, we have $\lambda_3(t_{j2})>0$ and I^s, K^s are well defined on path 2. We choose to disregard the borderline case where $\lambda_3(T)=0$; cf. (42c).

Appendix 5: Proof of Proposition 1

First consider *path 2* as the final path. Let t_{ij} be the coupling instant between path i and path j (such that path i precedes path j).

Path 4 → Path 2. On path 2, $G'(K)=a+r$; on path 4, $G'(K)$ increases since K decreases. Hence, for a coupling path 4 → path 2 it must hold that $G'(K)<a+r$ on path 4. This makes path 4 infeasible as an initial path [cf. (53)] and it must be preceded by some other path. Path 4 cannot be preceded by neither path 1 nor path 2 since on these paths we have $G'(K)>a+r$ and $G'(K)=a+r$, respectively. Could path 4 be preceded by path 3? Only if $G'(K)<a+r$ on path 3. But then the initial condition (53) cannot hold.

Path 3 → Path 2. The same conclusions as for path 4 → path 2 apply.

Path 1 → Path 2. This coupling is feasible. Notice that on path 1 it must hold that $\dot{K}>0$ since $G'(K)>a+r$ and $G'(K)$ must decrease to $G'(K)=a+r$. Hence, on path 1, $\dot{K}>0$ at least on some interval before the coupling instant t_{12}.

Now we have to check if path 2, 3, or 4 could precede the string path 1 → path 2. We will do this checking for the dividend policies of type 1, type 2, and type 3 [cf. (35)].

Path 4 → Path 1 → Path 2. Depending on the dividend policy we can have $C^*=0$ or $C^*=C^M$ at the coupling instant t_{41}.

$C^*(t_{41})=0$: From (5) we have $\dot{R}=(1-f)[G(K)-aK-rY]$. Since $Y=K-\bar{Z}-R$, and the state variables K and R are continuous, we have Y and $\dot{R}$ continuous. On path 4, $\dot{R}<0 \Rightarrow \dot{R}(t_{41}^-)<0$. But on path 1, $\dot{R}>0 \Rightarrow \dot{R}(t_{41}^+)>0$, which contradicts the continuity of $\dot{R}$. Notice that both paths 1 and 4 are boundary paths. We conclude that the coupling path 4 → path 1 is infeasible.

$C^*(t_{41}) = C^M$: For path 4 is holds that $\dot{R}(t_{41}^-) < 0$. Since $C^* = C^M$ across t_{41}, $\dot{R}$ will be continuous at $t = t_{41}$. Hence $\dot{R}(t_{41}^+) < 0$ must hold on path 1. On path 4 we have $I = 0 \Rightarrow \dot{K} = -aK < 0$, and on path 1 we have $\dot{K} = I^b - aK < 0$ for $t \to t_{41}$ (from the right) since $\dot{R}(t_{41}^+) < 0$. If $I^b > 0$, $\dot{K}$ will be discontinuous at t_{41}. Assume that $\dot{Y}$ is continuous. Then $\dot{K} = \dot{Y} + \dot{R}$ should be continuous too, which contradicts what has just been stated. Hence $\dot{Y}$ is discontinuous, but this contradicts that $\dot{Y}$ must be continuous since $\dot{Y} = k\dot{R}$. In conclusion, the coupling path 4 → path 1 is infeasible.

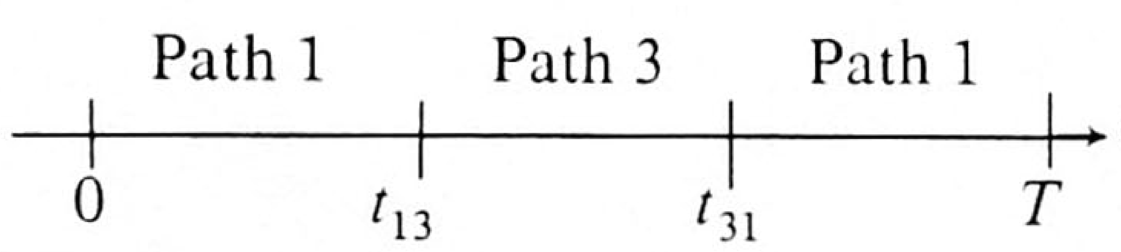
Path 3 → Path 1 → Path 2. Depending on the dividend policy we can have $C^*(t_{31}) = 0$ or $C^*(t_{31}) = C^M$.

$C^*(t_{31}) = 0$: On path 3 it holds that $\lambda_2 < 0$ and $\dot{\lambda}_2 = a\lambda_2 - (1-f)[G'(K) - a - r]\lambda_3$, whereas on path 1 we have $\dot{\lambda}_2 = \lambda_2 = 0$. Consider the coupling instant t_{31}. Since path 1 is singular, we must have $\lambda_3(t_{31}^+) > 0$ and $\dot{\lambda}_2(t_{31}^+) = \lambda_2(t_{31}^+) = 0$. Hence, on path 3 it must hold that

$$\dot{\lambda}_2(t_{31}^-) = -(1-f)[G'(K) - a - r]\lambda_3(t_{31}^-) \geq 0.$$

Thus, $G'(K) - a - r \leq 0$ on path 3 as $K \to K(t_{31}^-)$. However, on path 1 (which is to follow) $G'(K(t_{31}^+) - a - r > 0$ and to retain continuity of K it must hold that $G'(K(t_{31})) = a + r$. But $\dot{K} > 0$ on path 1, implying that $G'(K)$ become less than $a + r$ as t increases. This contradicts the requirement (52). Hence, the coupling path 3 → path 1 is infeasible.

$C^*(t_{31}) = C^M$: Consider the coupling instant t_{31} and recall that $(1+k)(\bar{Z} + R) - K \geq 0$ on path 3, whereas $(1+k)(\bar{Z} + R) - K = 0$ on path 1. If t_{31} is an entry point, then $(1+k)(\bar{Z} + R) - K > 0$ on some interval $(t_{31} - \varepsilon, t_{31})$, $\varepsilon > 0$.[12] But then path 3 violates the initial condition (53). Hence, if t_{31} is an entry point, we must have a path which precedes path 3. Such a path can only be path 1. Consider the following policy string.



At $t = 0$ it holds that $G'(K) > a + r$, and this can continue to hold on path 1 if K does not increase so much that $G'(K) \leq a + r$ occurs. To couple path 3 at t_{13} we must have $G'(K(t_{13}^+) \geq a + r$. Working backwards from $t = T$, we must have $G'(K(t_{31}^-) \leq a + r$ by the same argument as stated for the coupling path 3 → path 1 in the case where $C^*(t_{31}) = 0$. The contradiction now follows. Since $\dot{K} < 0$ on path 3, $G'(K)$ will increase as t increases from t_{13} to t_{31}.

[12]If t_{31} is not an entry point, $(1+k)(\bar{Z} + R) - K = 0$ on some interval $(t_{31} - \delta, t_{31})$. We return to this case later on.

It only remains to consider the policy string $1 \to 3 \to 1$, when t_{31} is not an entry point. Hence $(1+k)(\bar{Z}+R) - K = 0$ on some interval to the left of t_{31}. But then path 3 as well as path 1 are boundary arcs on an interval containing t_{31} and the arguments stated for the coupling path $4 \to$ path $1 \to$ path 2 ($C^* = C^M$) apply. In conclusion, the coupling path $3 \to$ path 1 is infeasible.

Path 2 → Path 1 → Path 2

$C^*(t_{21}) = 0$: On path 2, $G'(K) = a + r$ and $\dot{K} = 0$. On path 1, $G'(K) > a + r$ and $\dot{K} > 0$, implying that $G'(K)$ decreases. Obviously this coupling is infeasible.

$C^*(t_{21}) = C^M$: On paths 1 and 2 we have $\dot{\lambda}_2 = \lambda_2 = 0$. Hence,

$$0 = (1-f)(G'(K) - a - r)\lambda_3 \qquad \text{on path 2,}$$

$$0 = -(1-f)(G'(K) - a - r)\lambda_3 + \mu \quad \text{on path 1.} \tag{A.5}$$

Furthermore, $\lambda_3 > 0$ on path 1, $G'(K) = a + r$ on path 2, and $G'(K) > a + r$ on path 1. By continuity of K we must have $G'(K) = a + r$ on path 1 at $t = t_{21}$. But then $\mu(t_{21}^+) = 0$ in (A.5). Now, $\dot{R}(t_{21}^-) = 0$, and since $\dot{R}$ must be continuous across t_{21}, we must have $\dot{R}(t_{21}^+) = 0$. This implies $\dot{K}(t_{21}^+) = 0$ and $I^b = aK$ on path 1. Notice that $I = aK$ and $\dot{K}(t_{21}^-) = 0$ on path 2. Hence $I = aK$ is continuous across t_{21}. Extending by continuity of the control, $I = aK$ on an interval $[t_{21}, t_{21} + \varepsilon)$ implies $\dot{K} = 0$ and hence $G'(K) = a + r$. This, however, violates the condition $G'(K) > a + r$ on path 1. In conclusion, the coupling path $2 \to$ path 1 is infeasible.

We have now established that no path can precede the string path $1 \to$ path 2. Our analysis also shows that when we take path 1 as the final path, no paths can be coupled before path 1. Hence, we have only two candidates for an optimal policy, namely

(*) Path 1 → Path 2

(**) Path 1 throughout $[0, T]$. Q.E.D.

Appendix 6: Proof of Lemma 5

Define

$$A = -\frac{1}{(1-f)r} \ln\left(\frac{G(K^s) - aK^s - rY_{12}}{G(K^s) - aK^s}\right), \tag{A.6}$$

and note that $A > 0$. Eq. (54) can be written as

$$t_0 = t_{12} + A,$$

which yields

$$\frac{\mathrm{d}t_0}{\mathrm{d}r} = \frac{\mathrm{d}t_{12}}{\mathrm{d}r} + \frac{\mathrm{d}A}{\mathrm{d}r}. \tag{A.7}$$

First we note that

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{1}{(1-f)r^2}\left\{\ln\left(\frac{G(K^s) - aK^s - rY_{12}}{G(K^s) - aK^s}\right) + \frac{rY_{12}}{G(K^s) - aK^s - rY_{12}}\right\} \gtreqless 0,$$

for

$$\ln\left(\frac{G(K^s) - aK^s - rY_{12}}{G(K^s) - aK^s}\right) + \frac{rY_{12}}{G(K^s) - aK^s - rY_{12}} \gtreqless 0. \tag{A.8}$$

To simplify the notation define

$$\alpha = rY_{12} \quad \text{and} \quad \beta = G(K^s) - aK^s.$$

Hence (A.8) becomes

$$\ln\left(\frac{\beta - \alpha}{\beta}\right) + \frac{\alpha}{\beta - \alpha} \gtreqless 0 \Rightarrow \ln\left(1 - \frac{\alpha}{\beta}\right) \gtreqless \frac{-\alpha}{\beta - \alpha}. \tag{A.9}$$

Defining $z = \alpha/\beta$ yields in (A.9)

$$\ln(1 - z) \gtreqless \frac{-z}{1 - z} \Rightarrow \ln(1/(1 - z)) \lesseqgtr z/(1 - z). \tag{A.10}$$

Define

$$y = 1/(1 - z),$$

which yields in (A.10)

$$\ln y \lesseqgtr y - 1. \tag{A.11}$$

But $y - 1 > \ln y$ for all $y > 0$, except at $y = 1$ where $\ln y = y - 1$. However,

$y=1$ cannot occur since $y=1 \Rightarrow z=0 \Rightarrow \alpha=0 \Rightarrow r=0$ and/or $Y_{12}=0$. Comparing (A.8) with (A.11) we conclude that $\mathrm{d}A/\mathrm{d}r>0$.

The second step is to calculate $\mathrm{d}t_{12}/\mathrm{d}r$. On path 1, $\dot{K}>0$ implies $\dot{\mu}<0$ and μ decreases from $\mu(0)$ to zero at the start of path 2. The multiplier μ may jump, at $t=t_{12}$, from some positive $\mu(t_{12}^{+})$ to zero. From (48) we know that, whenever μ is differentiable, then

$$\dot{\mu}=(1-f)\dot{\lambda}_3(G'(K)-a-r)+(1-f)\lambda_3 G''(K)\dot{K},$$

and $\dot{\lambda}_3$ is given by (40c), i.e.,

$$\begin{aligned}\dot{\mu}=&(1-f)(G'(K)-a-r)(\mu(1+k)-(1-f)r\lambda_3)\\&+(1-f)\lambda_3 G''(K)\dot{K}.\end{aligned} \tag{A.12}$$

In (A.12) define

$$\phi=-(1-f)^2(G'(K)-a-r)r\lambda_3+(1-f)\lambda_3 G''(K)\dot{K},$$

$$\psi=(1-f)(G'(K)-a-r)(1+k),$$

and notice that $\phi<0$ and $\psi>0$ on path 1. Then (A.12) can be written

$$\dot{\mu}+\psi\mu=\phi. \tag{A.13}$$

By integration in (A.13) we obtain

$$\mu=\exp\left(-\int_0^t\psi(s)\,\mathrm{d}s\right)\left\{\mu(0)+\int_0^t\phi(s)\exp\left(\int_0^s\psi(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}. \tag{A.14}$$

At $t=t_{12}$ we have

$$\begin{aligned}\mu(t_{12}^{+})=&\exp\left(-\int_0^{t_{12}}\psi(s)\,\mathrm{d}s\right)\left\{\mu(0)+\int_0^{t_{12}}\phi(s)\exp\left(\int_0^s\psi(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}\\&\geq 0.\end{aligned} \tag{A.15}$$

From (A.5) (and the remarks below that equation) we know that $\mu(t_{12}^{+})=0$ and (A.15) holds with equality. Let

$$F(t_{12},r)=\mu(0)+\int_0^{t_{12}}\phi(s)\exp\left(\int_0^s\psi(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s=0, \tag{A.16}$$

and recall that ψ and ϕ depend on r. Hence, by (A.16), t_{12} is implicitly given as a function of r. By the implicit function theorem we obtain

$$\frac{\mathrm{d}t_{12}}{\mathrm{d}r} = -\frac{\partial F/\partial r}{\partial F/\partial t_{12}}$$

$$= -\frac{\displaystyle\int_0^{t_{12}}\left\{\phi(s)\exp\left(\int_0^s \psi(\tau)\,\mathrm{d}\tau\right)\int_0^s \frac{\partial \psi}{\partial r}\,\mathrm{d}\tau + \exp\left(\int_0^s \psi(\tau)\,\mathrm{d}\tau\right)\frac{\partial \phi(s)}{\partial r}\right\}\mathrm{d}s}{\displaystyle\phi(t_{12})\exp\left(\int_0^{t_{12}}\psi(\tau)\,\mathrm{d}\tau\right)}$$

$$= -1\Big/\left(\phi(t_{12})\exp\left(\int_0^{t_{12}}\psi(\tau)\,\mathrm{d}\tau\right)\right)$$

$$\times \int_0^{t_{12}}\exp\left(\int_0^s \psi(\tau)\,\mathrm{d}\tau\right)\{-\phi(s)(1-f)(1+k)s$$

$$-(1-f)(G'(K)-a-r)(1-f)\lambda_3 + (1-f)^2 r\lambda_3\}\,\mathrm{d}s. \tag{A.17}$$

In (A.17) we have $\phi(t_{12}) < 0$, and

$$\phi(s)(1+k)s + (1-f)(G'(K)-a-r)\lambda_3 - (1-f)r\lambda_3 < 0 \tag{A.18}$$

is sufficient for $\mathrm{d}t_{12}/\mathrm{d}r > 0$. But $G'(K(t_{12}^+)) = a + r$ must hold to guarantee continuity of K [cf. the remarks below (A.5)]. Then (A.18) holds since $\phi < 0$.

We have shown that $\mathrm{d}A/\mathrm{d}r > 0$, $\mathrm{d}t_{12}/\mathrm{d}r > 0$, and using (A.7) completes the proof. Q.E.D.

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